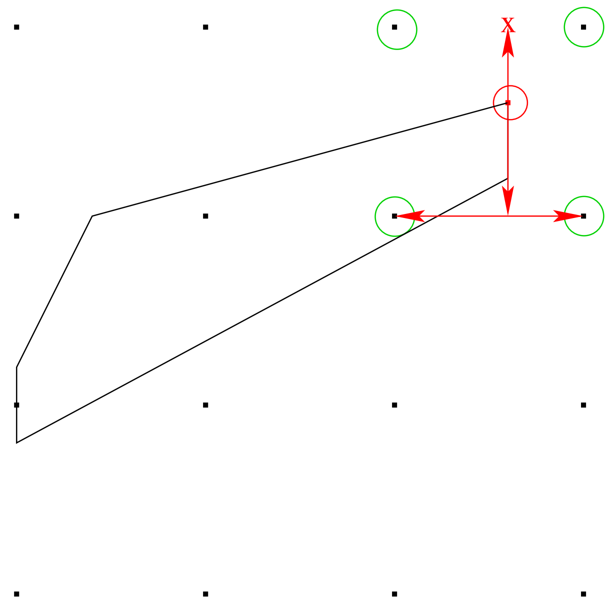
Integer Programming

Lecture 23

Heuristics in Integer Programming

- Heuristic methods are an extremely important aspect of integer programming in practice.
- Often it is the case that a near-optimal solution is "good enough."
- Furthermore, even if an optimal solution is required, heuristic methods can accelerate the solution process.
- Heuristic methods are generally used in one of two modes.
  - As a stand-alone procedure used directly to obtain a solution or as a means to obtain an initial bound (*metaheuristics*).
  - As an integrated part of a branch-and-bound procedure (*primal heuristics*).
- In this lecture, we will focus on the latter use, since this is the way in which heuristics are generally used in off-the-shelf solvers.

Simple Rounding

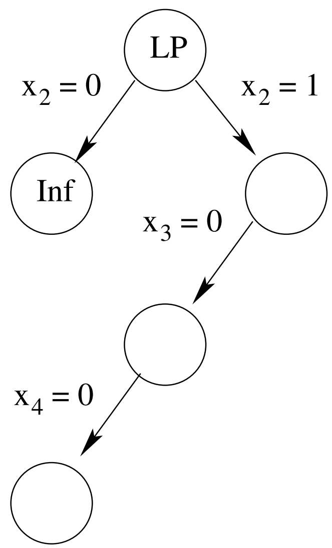


1. We first solve the LP relaxation to obtain an (infeasible) solution.
2. There may be a number of integer variables with fractional values.
3. We can round these variables one at a time, but there is no way to guarantee that this will lead to a feasible solution.
4. If there are k such variables, there are 2^k ways of rounding.
5. Use backtracking

Backtracking example

minimize x_1
subject to:
$x_1 - 2x_2 + 2x_3 + 2x_4 = -1$
$x_1 \geq 0$
$x_2, x_3, x_4 \in \{0, 1\}$

$$\hat{x} = (0, 0.5, 0, 0)$$



When an LP is solved after each fixing: Diving (Bixby et al., 2000)

Rounding

- Other variants

  1. Randomized rounding may help in specific contexts: single machine scheduling, set covering, set packing etc. (Bertsimas and Weismantel, 2005)
  2. Rounding problem can be explicitly stated as a binary program (Berthold 2006)

- Importance of rounding

  1. Rounding is cheap
  2. Many different variants of rounding may be deployed easily
  3. Rounding is an important step in several other heuristics

Feasibility Pump: The Basic Scheme

- We start from any $\hat{x}^0 \in \mathcal{P}$, and round to obtain $\widetilde{x}^0$.
- We look for a point $\hat{x}^1 \in \mathcal{P}$ which is *as close as possible* to $\widetilde{x}^0$ by solving the problem:

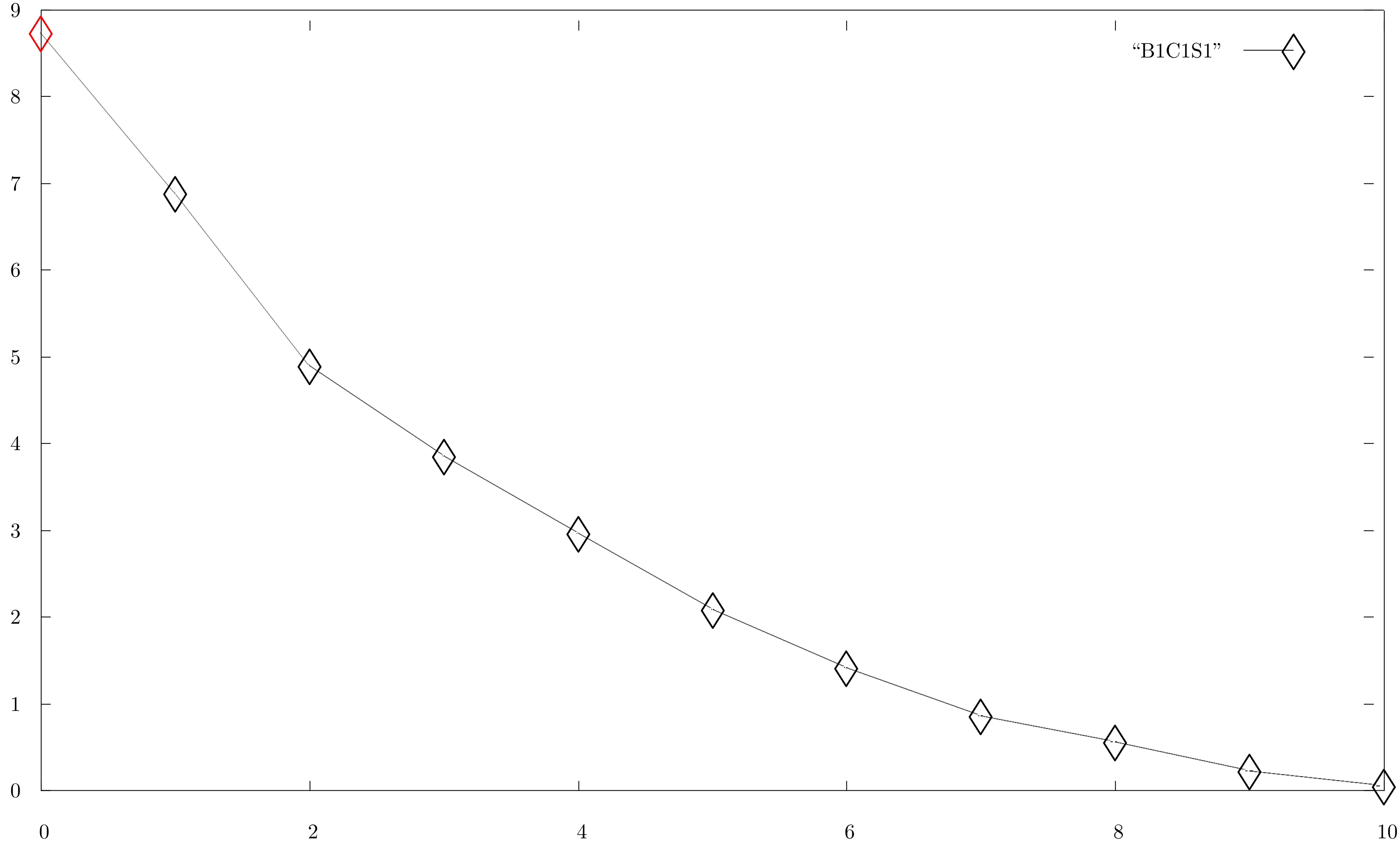
$$\min\{\Delta(x, \widetilde{x}) \mid x \in \mathcal{P}\}$$

  If we choose the measure $\Delta(x, \widetilde{x})$ properly, this problem is easily solvable.

- If $\hat{x}^1 \in \mathcal{S}$, we are done.
- Otherwise, we obtain $\widetilde{x}^1$ by rounding $\hat{x}^1$, and repeat.
- From a geometric point of view, this simple heuristic generates *two hopefully convergent trajectories of points $\hat{x}^i$ and $\widetilde{x}^i$*.
- These satisfy feasibility in a complementary but partial way:

1. $\hat{x}^i$, satisfies the linear constraints,
2. $\widetilde{x}^i$, the integrality requirements.

FP: Plot of the infeasibility measure $\Delta(\hat{x}^i, \widetilde{x}^i)$ at iteration i



FP: Definition of $\Delta(\hat{x}, \widetilde{x})$

- We consider the L_1-*norm distance* between a vector $x \in \mathcal{P}$ and a vector $\widetilde{x} \in \mathcal{S}$:

$$\Delta(x, \widetilde{x}) = \sum_{j \in I} |x_j - \widetilde{x}_j|$$

  where I is the set of indices of the integer variables.

- The *continuous variables* do not contribute to this function.

- In the case of a *binary* MILP:

$$\Delta(x, \widetilde{x}) := \sum_{j \in I : \widetilde{x}_j = 0} x_j + \sum_{j \in I : \widetilde{x}_j = 1} (1 - x_j)$$

- Given an integer $\widetilde{x}$, the *closest point* $\hat{x} \in \mathcal{P}$ can therefore be determined *by solving the LP*:

$$\min\{\Delta(x, \widetilde{x}) : Ax \le b\}$$

FP: Implementation

- We can think of the distance as a pressure difference between $\hat{x}$ and $\widetilde{x}$ that we try to reduce by *pumping* the integrality of $\widetilde{x}$ into $\hat{x}$.
- On the other hand, it is clearly a measure of vicinity and therefore defines a neighborhood.
- The main problem with this method is stalling when $\Delta(\hat{x}, \widetilde{x})$ stops decreasing (we may produce the same solution).
  - In this case, we *reverse the rounding* of *some* variables $\hat{x}_j, \ j \in I$, even if this increases $\Delta(\hat{x}, \widetilde{x})$
  - This is done so as to minimize the increase in the current value of $\Delta(\hat{x}, \widetilde{x})$.

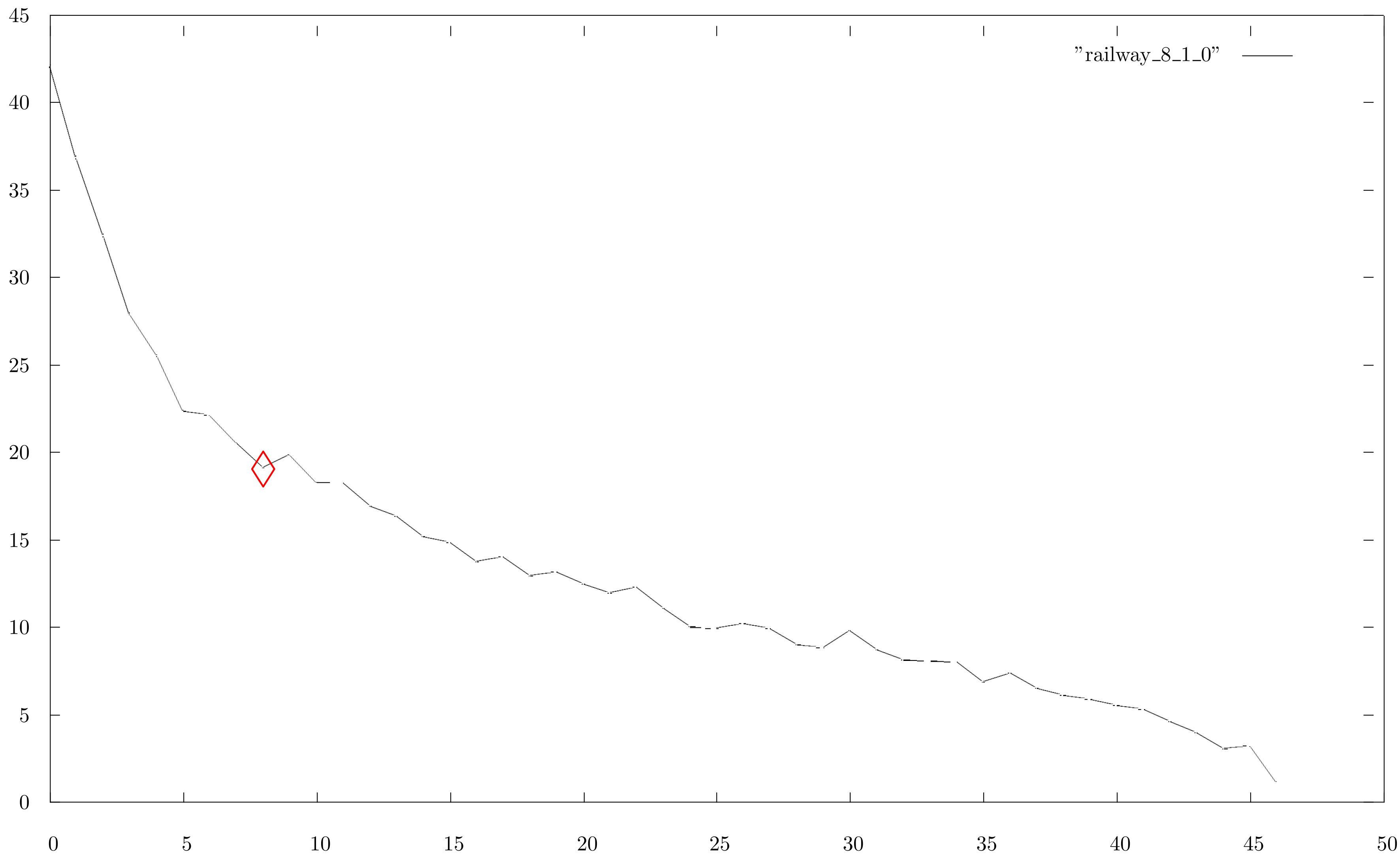
FP: A first implementation

1. initialize nIT := 0 and $\hat{x} :=$ argmax$\{c^\top x : Ax \le b\}$;
2. if $\hat{x}$ is integer, return($\hat{x}$);
3. let $\widetilde{x} := [\hat{x}]$ (= rounding of $\hat{x}$);
4. while (time < TL) do
5. let nIT := nIT $+ 1$ and $\hat{x} :=$ argmin$\{\Delta(x, \widetilde{x}) : Ax \le b\}$;
6. if $\hat{x}$ is integer, return($\hat{x}$);
7. if $\exists\, j \in I : [\hat{x}_j] \ne \widetilde{x}_j$ then
8. $\widetilde{x} := [\hat{x}]$

   else

9. flip the rand(T/2,3T/2) entries $\widetilde{x}_j$ with max $|\hat{x}_j - \widetilde{x}_j|$
10. endif
11. enddo

FP: Plot of the infeasibility measure $\Delta(\hat{x}, \widetilde{x})$ at each pumping cycle



Neighborhood Search

- Rounding schemes explore neighborhoods defined by $\lfloor x_i^* \rfloor \leq x_i \leq \lceil x_i^* \rceil, i \in I$.
- Feasibility pump explores neighborhoods defined by the *nearby* basic feasible solutions
- Pivoting heuristics explores neighborhoods of $\hat{x}$ defined by the respective pivoting and complementing schemes.

Each of the above neighborhoods are explored using special methods

Exploring Neighborhoods

- The MILP solver itself can also be used as a search tool!
- A small neighborhood expressed as a MILP can be explored by using a MILP solver over it.
- Recall the "optimal rounding problem."

Local Branching

- Now assume we have a feasible solution $\bar{x}$, the so-called reference solution, and let $\overline{S} := \{j \in I \mid \bar{x}_j = 1\}$ denote the binary support of $\bar{x}$.
- For a given positive integer parameter k, we define the k-OPT neighborhood $\mathcal{N}(\bar{x}, k)$ of $\bar{x}$ as the set of the feasible solutions satisfying

$$\Delta(x, \bar{x}) := \sum_{j \in \overline{S}} (1 - x_j) + \sum_{j \in I \setminus \overline{S}} x_j \le k, \quad (1)$$

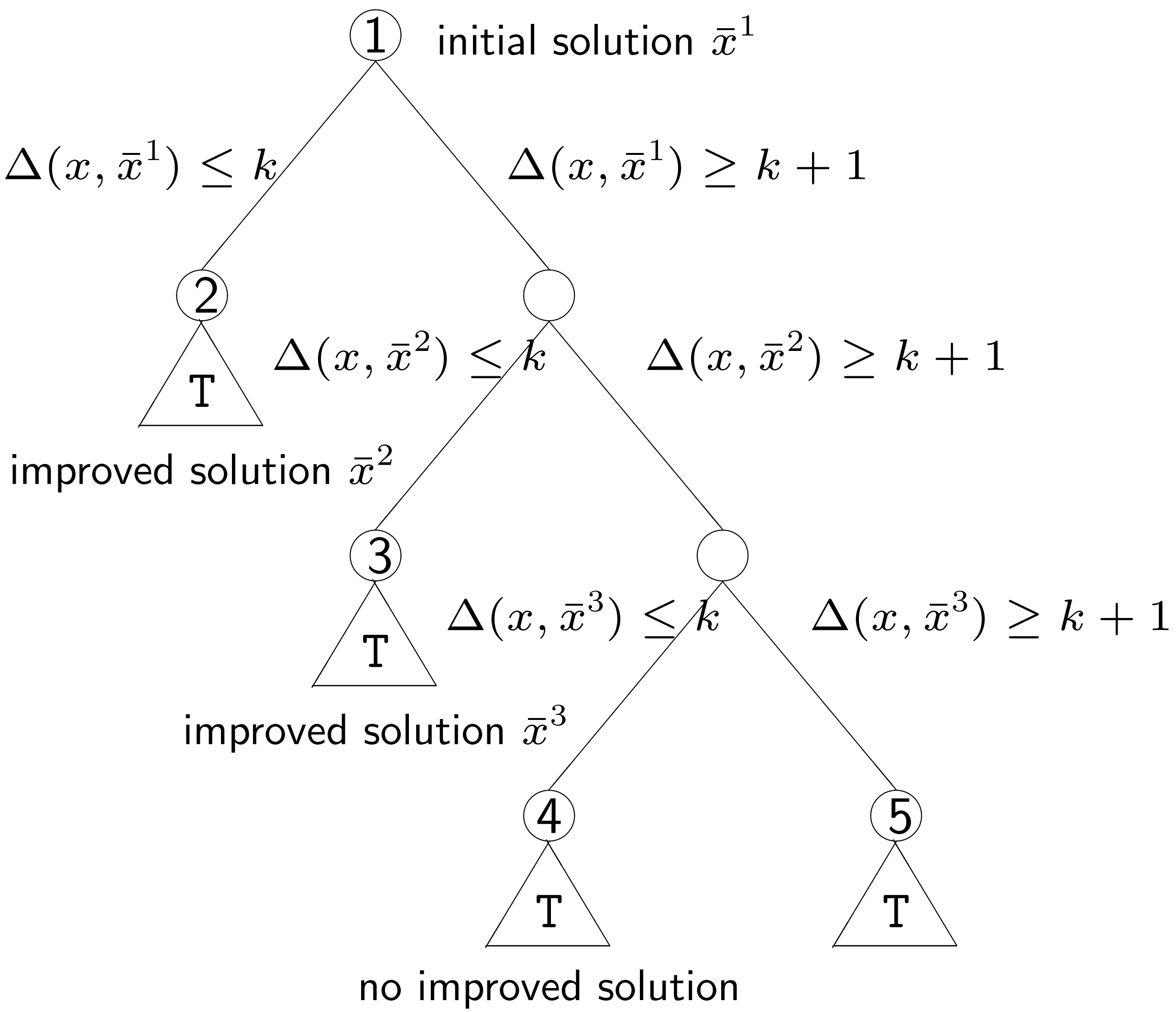
known as the *local branching constraint*.

- This constraint requires at most k variables have values different from $\bar{x}$.
- Constraint (1) can also be used to branch within a branch and bound:

$\Delta(x, \bar{x}) \le k$ (left branch) **or** $\Delta(x, \bar{x}) \ge k + 1$ (right branch)

- The neighborhoods defined by the local branching constraints can be searched by using a MILP solver recursively.

LB: The Basic Scheme



LB: Enhancements

- The previous scheme can be enhanced in two ways:
  - Imposing a time/node limit on the *left-branch* nodes:
    - In some cases, the exact solution of the left-branch node can be too time consuming for the value of the parameter k at hand.
    - Hence, from the point of view of a heuristic, it is reasonable to impose a time/node limit for the left-branch computation.
  - Increasing diversification:
    - A further improvement of the heuristic performance can be obtained by exploiting diversification mechanisms in the spirit of metaheuristic techniques.
    - In this scheme, diversification is applied by *varying the value of* k and accepting non-improving solutions.
- On the other hand, it is easy to see that an alternative implementation would be within the branch-and-cut tree of a MILP solver.
- More precisely, we search using the branch-and-cut algorithm itself for a *fixed number of nodes*.
- Whenever a new incumbent has been found, this LB can be fed into this local search to limit enumeration.

Relaxation Induced Neighborhood Search

- A similar concept of neighborhood takes into account simultaneously both

  - the *incumbent* solution $\bar{x}$, and
  - the *the solution of the continuous relaxation* $\hat{x}$,

  at a given node of the branch-and-bound tree.

- $\bar{x}$ and $\hat{x}$ are compared and all the binary variables that assume the same value are *hard-fixed* in an associated MILP.

- This associated MILP is then solved by using the MILP solver as a black-box.

- In case the incumbent solution is improved, $\bar{x}$ is updated in the rest of the tree.

- This method turns out to give very competitive results on general MILPs.

- It is particularly suitable in the scheduling context where the problem is very constrained and any non-trivial value of k would be too large.

RINS Formulation

RINS: Let $\tilde{x}$ be a known feasible solution and let $\hat{x}$ be an LP-solution at some node in the search tree. We create a new MILP (Danna et al., 2005):

$$\text{minimize } z = cx$$

s.t.

$$Ax \leq b,$$

$$x_i = \tilde{x}_i \quad \forall i \quad \text{s.t. } \tilde{x}_i = \hat{x}_i$$

$$x \in \mathbb{Z}^r \times \mathbb{R}^{n-r}.$$

Working with Infeasible Solutions

- Sometimes waiting to have a fully feasible solution before starting a local search approach is unnecessary.
- Combining the work of both *FP and LB* provides the following more flexible scheme:

1. FP is executed for a *limited number of iterations* and the *integer (infeasible)* solution $\widetilde{x}$ with minimum distance Δ to a feasible solution $\hat{x}$ of the LP relaxation is stored;
2. LB starts by using $\widetilde{x}$ *as a reference solution*, replacing the original objective function with

$$\min \sum_{i \in T} y_i$$

where T is the set of the indices of the constraints violated by $\widetilde{x}$ and a binary variable y_i has been defined for each constraint $i \in T$.

Working with Infeasible Solutions (cont.)

- Hence, in a first phase, LB attempts to improve the current infeasible solution by reducing the number of infeasible constraints in the spirit of the first phase of the simplex algorithm.
- In the second phase, once a feasible solution has been found, the original objective function is then restored and LB takes care of *improving the quality* of such a solution.