Integer Programming

Lecture 7

Computational Discrete Optimization

- Before going any deeper into the theory of integer optimization, we now delve into how integer optimization problems are solved in practice.
- In this lecture, we introduce *branch and bound*, the most widely used algorithmic framework for solving MILPs in practice.
- Branch and bound is not so much a complete algorithm as a *framework*.
- A particular implementation consists of a collection of specific decision-making procedures bound together by a control mechanism.
- A wide variety of different algorithms can be obtained by implementing the constituent procedures in different ways.
- The most fundamental constituent procedures are
  - A method for obtaining upper and lower *bounds* on the value of the optimal solution (usually by solving a *relaxation* or *dual*); and
  - A method for producing a *valid disjunction* violated by a given (infeasible) solution.
- In the next few lectures, we will examine the details of how these types of procedures can be implemented.

Branch and Bound

- *Branch and bound* is the most widely used algorithmic framework for solving MILPs.
- It is a recursive, divide-and-conquer approach.
- Suppose $\mathcal{S}$ is the feasible set for an MILP and we wish to compute $\max_{x \in \mathcal{S}} c^\top x$.
- Consider a partition of $\mathcal{S}$ into subsets $\mathcal{S}_1, \ldots \mathcal{S}_k$. Then

$$\max_{x \in \mathcal{S}} c^\top x = \max_{1 \leq i \leq k} \max_{x \in \mathcal{S}_i} c^\top x$$

- In other words, we can optimize over each subset separately.
- Idea: If we can't solve the original problem directly, we might be able to solve the smaller *subproblems* recursively.
- Dividing the original problem into subproblems is called *branching*.
- Taken to the extreme, this scheme is equivalent to complete enumeration.

A Generic Branch-and-Bound Algorithm

Add root optimization problem $\mathcal{S}_0 := \mathcal{S}$ to a priority queue Q.
Set global upper bound $U \leftarrow \infty$ and global lower bound $L \leftarrow -\infty$
Set $T := \emptyset$ (the set of terminal nodes).
while $U > L$ **do**
    Remove the highest priority subproblem $\mathcal{S}_i$ from Q.
    Bound $\mathcal{S}_i$ to obtain upper bound $U(i)$ and lower bound $L(i)$.
    if $U(i) > L$ **then**
        Branch to create child subproblems $\mathcal{S}_{i_1}, \ldots, \mathcal{S}_{i_k}$ of subproblem $\mathcal{S}_i$ by partitioning $\mathcal{S}_i$
        Add $\mathcal{S}_{i_1}, \ldots, \mathcal{S}_{i_k}$ to Q with initial bounds $U(i_j) = U(i)$ and $L(i_j) = -\infty$ for $1 \leq j \leq k$.
    else
        Add $\mathcal{S}_i$ to T.
    end if
    Set $U \leftarrow \max_{k \in Q \cup T} U(k)$.
    Set $L \leftarrow \max\{L(i), L\}$.
end while

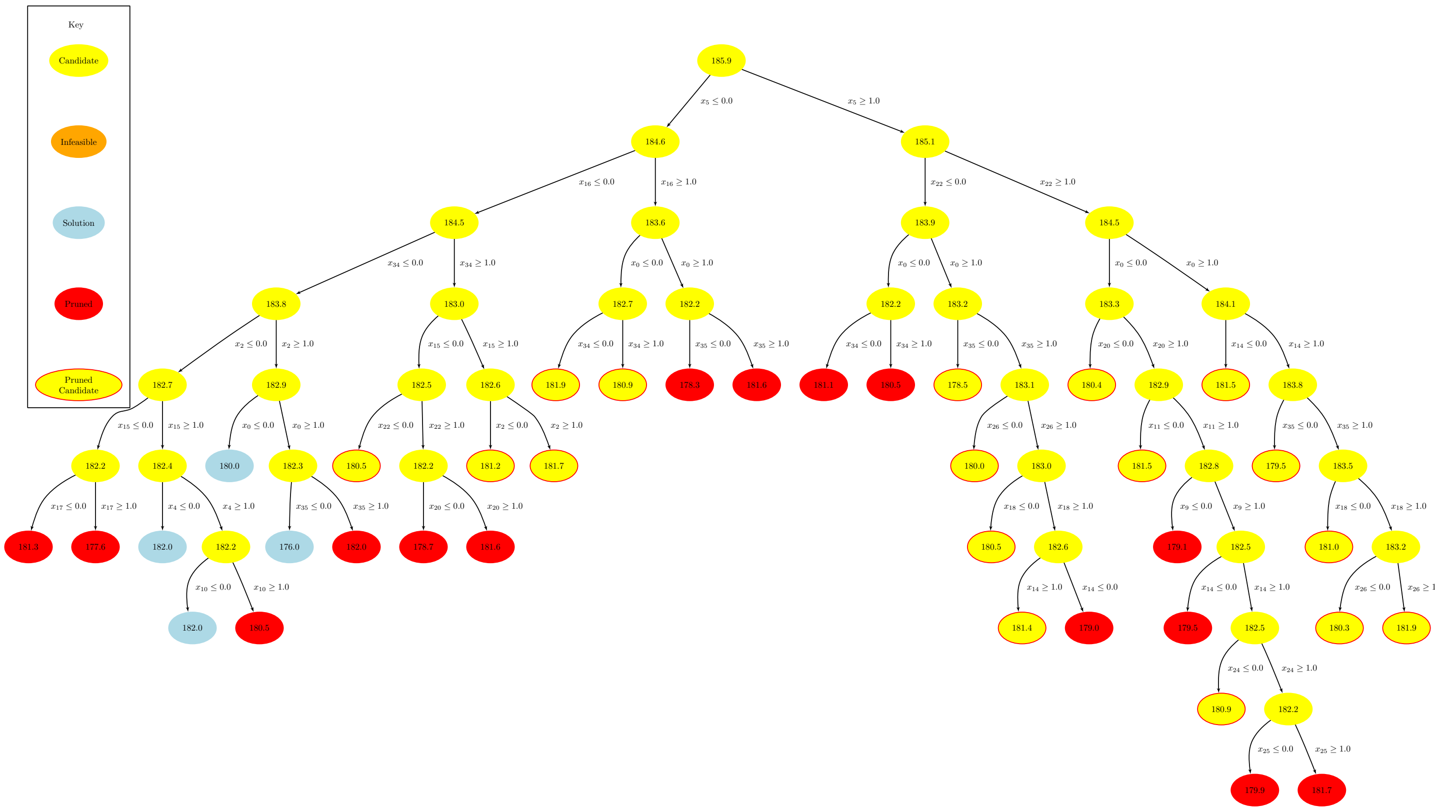
Missing Pieces

- What is the "priority" by which the subproblems are ordered?
- How do we get the upper and lower bounds?
- How do we partition a given subproblem?
- Is this algorithm guaranteed to terminate?
- Will it produce the optimal solution?
- Is the algorithm "efficient"?

Visualizing Branch and Bound

- It will be useful to be able to visualize the evolution of the branch-and-bound algorithm.
- Due to the recursive nature of the algorithm, the collection of subproblems produced can be thought of as forming a *branch-and-bound tree*.
- Each subproblem is connected to
  - its *parent*, the subproblem that was partitioned to yield it, and
  - its *children*, the subproblems resulting from further partitioning.
- The algorithm evolves by searching this dynamically generated tree.
- The search inevitably involves many dead ends and efficiency is improved by avoiding as many of them as possible.
- For theoretical reasons, it is conjectured that there is no way to completely avoid such dead ends.

Branch and Bound Tree



The Gap

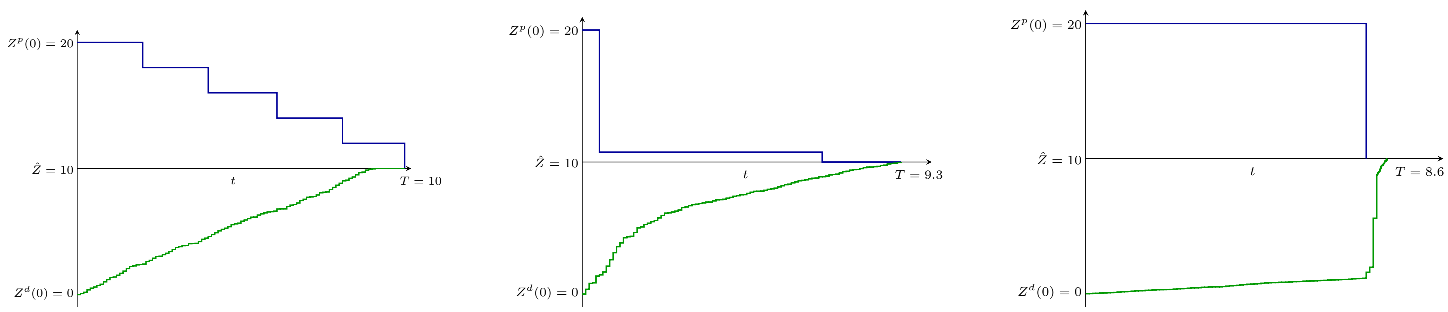
- Throughout the algorithm, we maintain a global upper bound U and a global lower bound L.
  - The lower bound comes from the *current incumbent* (the best feasible solution found so far).
  - The upper bound is that of the candidate node with the best bound.
- Optimality of the current incumbent is theoretically proved when $U = L$, but we usually terminate when $Q = \emptyset$ (this guarantees $U = L$).
- As the algorithm proceeds, the *relative optimality gap*

$$\frac{|U - L|}{\max\{|L|, |U|\}}$$

  (or simply *the gap*) gives us a quality guarantee for the incumbent.
- Even when branch-and-bound terminates early (due to time constraints), it provides this guarantee.
- This is what makes the method *exact* (as opposed to heuristic).

Evolution of the Algorithm

- As the algorithm proceeds, the gap decreases until reaching zero.
- The goal of the algorithm is to decrease this gap as quickly as possible.
- Decreasing the gap involves improving both the upper and lower bounds, which introduces important tradeoffs.
- It is tempting to view the current gap or its evolution as an indication of progress, but its predictive power is limited.



(a) (b) (c)

Figure 1: Evolution of the gap

Importance of Disjunction

- As we know, the difficulty in solving an integer optimization problem arises from the requirement that certain variables take on integer values.
- Such requirements are equivalent to logical *disjunctions*, constraints of the form
$$x \in \bigcup_{1 \leq i \leq k} X_i$$
for $X_i \subseteq \mathbb{R}^n, i \in 1, \ldots, k$.
- The integer variables in a given formulation may represent logical conditions that were originally expressed in terms of disjunction.
- In fact, the MILP Representability Theorem tells us that the feasible region of any MILP can be expressed in the form
$$\mathcal{S} = \bigcup_{i=1}^{k} \mathcal{P}_i + \mathrm{intcone}\{r^1, \ldots, r^t\},$$
for some appropriately chosen polytopes $\mathcal{P}_1, \ldots, \mathcal{P}_k$ and vectors $r^1, \ldots, r^t \in \mathbb{Z}^n$ (recall the proof that $\mathrm{conv}(\mathcal{S})$ is a rational polyhedron).

Two Conceptual Reformulations

- From what we have seen so far, we have two conceptual reformulations of a given integer optimization problem.
- The first is in terms of *disjunction*:

$$\max \left\{ c^\top x \mid x \in \left(\bigcup_{i=1}^{k} \mathcal{P}_i + \text{intcone}\{r^1, \ldots, r^t\} \right) \right\} \qquad \text{(DIS)}$$

- The second is in terms of *valid inequalities*:

$$\max \left\{ c^\top x \mid x \in \text{conv}(\mathcal{S}) \right\} \qquad \text{(CP)}$$

  where $\mathcal{S}$ is the feasible region.

- In principle, if we had a method for generating either of these reformulations, this would lead to a practical method of solution.
- Unfortunately, these reformulations are necessarily of exponential size in general, so there can be no way of generating them efficiently.

Valid Disjunctions

- In practice, we dynamically generate parts of the reformulations (CP) and (DIS) in order to obtain a proof of optimality for a particular instance.
- We can think of the concept of a *valid inequality* as arising from our desire to approximate $\mathrm{conv}(\mathcal{S})$ (the feasible region of (CP)).
- Similarly, we also have the concept of *valid disjunction*, arising from a desire to approximate the feasible region of (DIS).

Definition 1. *Let* $\{X_i\}_{i=1}^k$ *be a collection of subsets of* $\mathbb{R}^n$. *Then if* $\bigcup_{1 \leq i \leq k} X_i \supseteq \mathcal{S}$, *the disjunction associated with* $\{X_i\}_{i=1}^k$ *is said to be* valid *for an MILP with feasible set* $\mathcal{S}$.

Definition 2. *If* $\{X_i\}_{i=1}^k$ *is a disjunction valid for* $\mathcal{S}$ *and* X_i *is polyhedral for all* $i \in \{1, \ldots, k\}$, *then we say the disjunction is* linear.

Definition 3. *If* $\{X_i\}_{i=1}^k$ *is a disjunction valid for* $\mathcal{S}$ *and* $X_i \cap X_j = \emptyset$ *for all* $i, j \in \{1, \ldots, k\}$, *we say the disjunction is* partitive.

Definition 4. *If* $\{X_i\}_{i=1}^k$ *is a disjunction valid for* $\mathcal{S}$ *that is both linear and partitive, we call it* admissible.

Branching in Branch and Bound

- Branching is achieved by selecting an admissible disjunction $\{X_i\}_{i=1}^k$ and using it to partition $\mathcal{S}$, e.g., $\mathcal{S}_i = \mathcal{S} \cap X_i$.
- We only consider linear disjunctions so that the subproblem remain MILPs after branching.
- The reason for choosing partitive disjunctions is self-evident.
- The way this disjunction is selected is called the *branching method* and is a topic we will examine in some depth.
- Generally speaking, we want $x^* \notin \cup_{1 \leq i \leq k} X_i$, where x^* is the (infeasible) solution produced by solving the *bounding problem*.
- In this case, we say the disjunction is *violated* by x^*.
- A typical disjunction is

$$X_1 = \{x \in \mathbb{R}^n \mid x_j \leq \lfloor x_j^* \rfloor\}, \tag{1}$$
$$X_2 = \{x \in \mathbb{R}^n \mid x_j \geq \lceil x_j^* \rceil\}, \tag{2}$$

where $x^* \in \operatorname{argmax}_{x \in \mathcal{P}} c^\top x$.

Bounding in Branch and Bound

- The *bounding problem* is a problem solved to obtain a bound on the optimal solution value of a subproblem $\max_{x \in \mathcal{S}_i} c^\top x$.
- Typically, the bounding problem is either a relaxation or a dual of the subproblem (these concepts will be defined formally in Lecture 8).
- Solving the bounding problem serves two purposes.
  - In some cases, the solution x^* to the relaxation may actually be a feasible solution ($x^* \in \mathcal{S}$), in which case $c^\top x^*$ is a *global lower bound*.
  - *Bounding* enables us to inexpensively compute a bound $U(i)$ on the optimal solution value of subproblem i.
- If $U(i) \leq L$, then $\mathcal{S}_i$ can't contain a solution strictly better than the best one found so far.
- Thus, we may discard or *prune* subproblem i.

Constructing a Bounding Problem

- There are many ways to construct a bounding problem and this will be the topic of later lectures.
- The easiest of the these is to form the *LP relaxation* $\max_{\mathcal{P} \cap \mathbb{R}^n_+ \cap X_i}$, obtained by dropping the integrality constraints.
- For the rest of the lecture, assume all variables have finite upper and lower bounds.

LP-based Branch and Bound: Initial Subproblem

- In LP-based branch and bound, we first solve the LP relaxation of the original problem. The result is one of the following:

  1. The LP is infeasible $\Rightarrow$ MILP is infeasible.
  2. We obtain a feasible solution for the MILP $\Rightarrow$ optimal solution.
  3. We obtain an optimal solution to the LP that is not feasible for the MILP $\Rightarrow$ upper bound.

- In the first two cases, we are finished.
- In the third case, we must branch and recursively solve the resulting subproblems.

Branching in LP-based Branch and Bound

- In LP-based branch and bound, the most commonly used disjunctions are the *variable disjunctions*, imposed as follows:
  - Select a variable i whose value $\hat{x}_i$ is fractional in the LP solution.
  - Create two subproblems.
    - In one subproblem, impose the constraint $x_i \leq \lfloor \hat{x}_i \rfloor$.
    - In the other subproblem, impose the constraint $x_i \geq \lceil \hat{x}_i \rceil$.
- What does it mean in a 0–1 problem (problem for which all variables take on only values 0 or 1)?

The Geometry of Branching

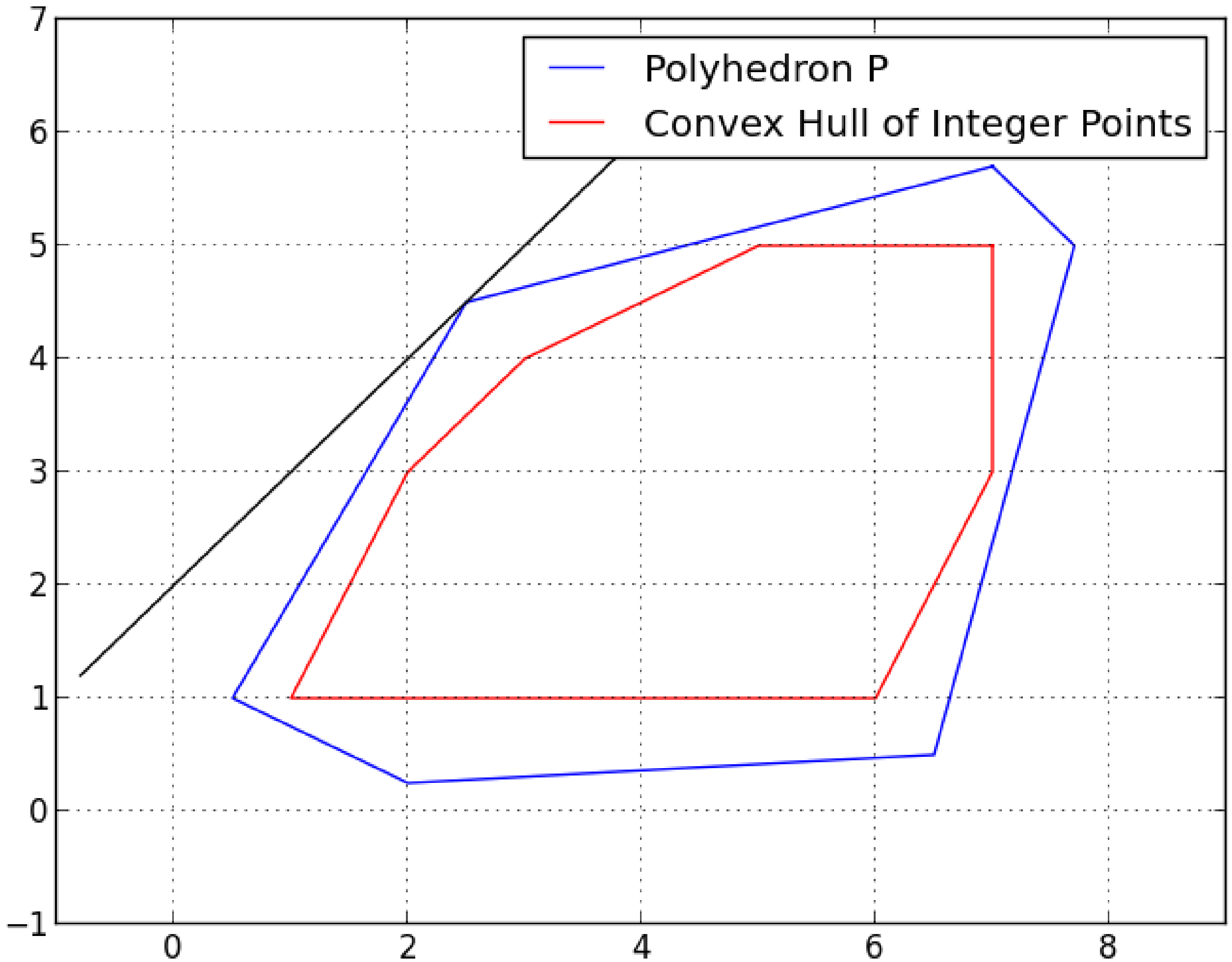


Figure 2: The original feasible region

The Geometry of Branching (cont'd)

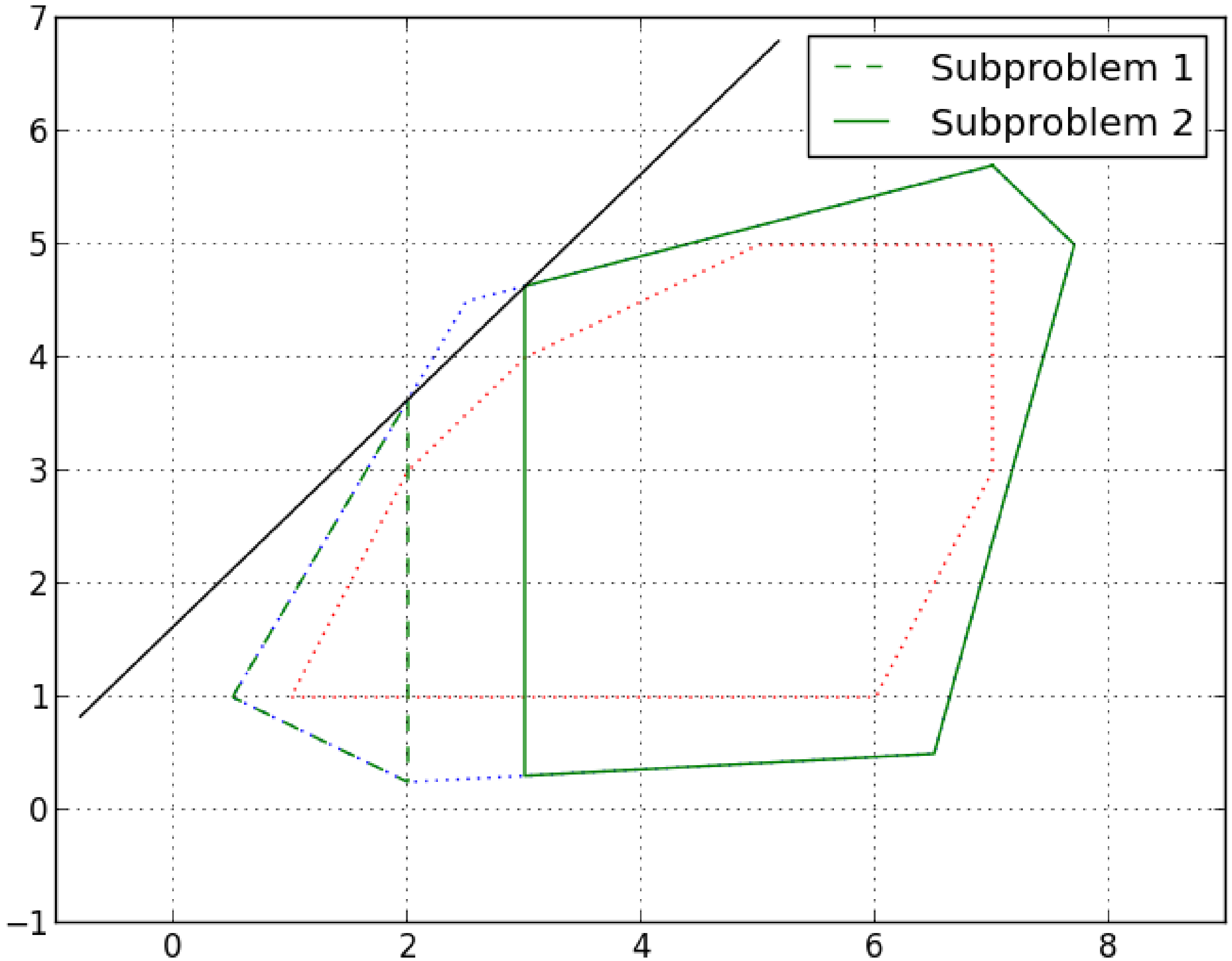


Figure 3: Branching on disjunction $x_1 \leq 2$ OR $x_1 \geq 3$

Continuing the Algorithm After Branching

- After branching, we solve each of the subproblems recursively.
- Now we have an additional factor to consider.
- As mentioned earlier, if the optimal solution value to the LP relaxation is smaller than the current lower bound, we need not consider the subproblem further.
- This is the key to the efficiency of the algorithm.
- Terminology
  - If we picture the subproblems graphically, they form a *search tree*.
  - Each subproblem is linked to its *parent* and eventually to its *children*.
  - Eliminating a problem from further consideration is called *pruning*.
  - The act of bounding and then branching is called *processing*.
  - A subproblem that has not yet been considered is called a *candidate* for processing.
  - The set of candidates for processing is called the *candidate list*.

The Geometry of Branching

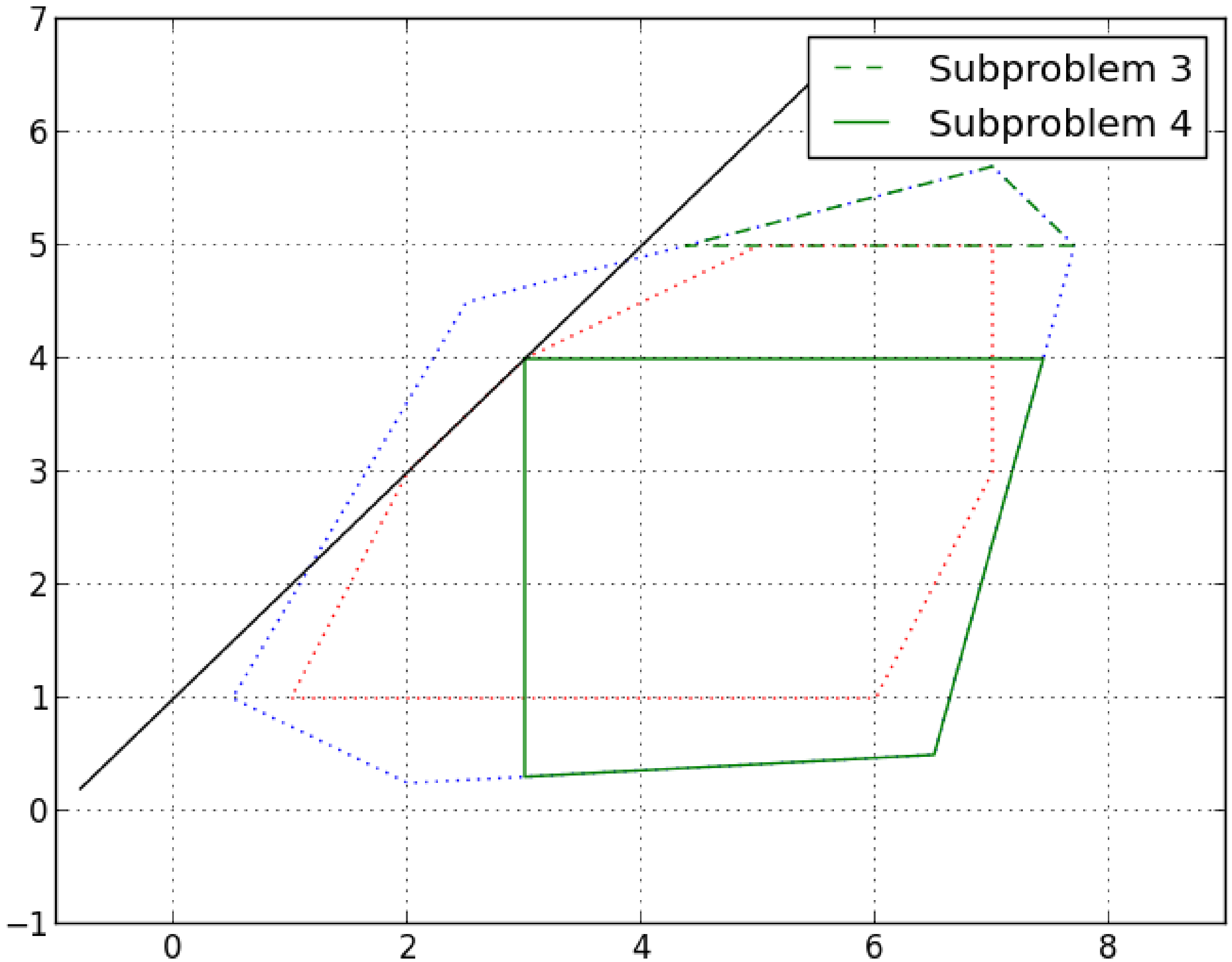


Figure 4: Branching on disjunction $x_2 \leq 4$ OR $x_2 \geq 5$ in Subproblem 2

LP-based Branch and Bound Algorithm

1. To start, derive a lower bound L using a heuristic method.
2. Put the original problem on the candidate list.
3. Select a problem $\mathcal{S}_i$ from the candidate list and solve the LP relaxation to obtain the bound $U(i)$.
   - If the LP is infeasible $\Rightarrow$ node can be pruned.
   - Otherwise, if $U(i) \leq L \Rightarrow$ node can be pruned.
   - Otherwise, if $U(i) > L$ and the solution is feasible for the MILP $\Rightarrow$ set $L \leftarrow U(i)$.
   - Otherwise, branch and add the new subproblem to the candidate list.
4. If the candidate list in nonempty, go to Step 2. Otherwise, the algorithm is completed.

Algorithmic Choices in Branch and Bound

- Although the basic algorithm is straightforward, the efficiency of it in practice depends strongly on making good algorithmic choices.
- These algorithmic choices are made largely by heuristics that guide the algorithm.
- Basic decisions to be made include
  - The bounding method(s).
  - The method of selecting the next candidate to process.
    - "Best-first" chooses the candidate with the highest upper bound.
    - Under mild conditions, this rule minimizes the size of the tree (why?).
    - There may be practical reasons to deviate from this rule.
  - The method of branching.
    - Branching wisely is extremely important.
    - A "poor" branching decision can slow the algorithm significantly.
- We will cover the last two topics in more detail in later lectures.

A Thousand Words

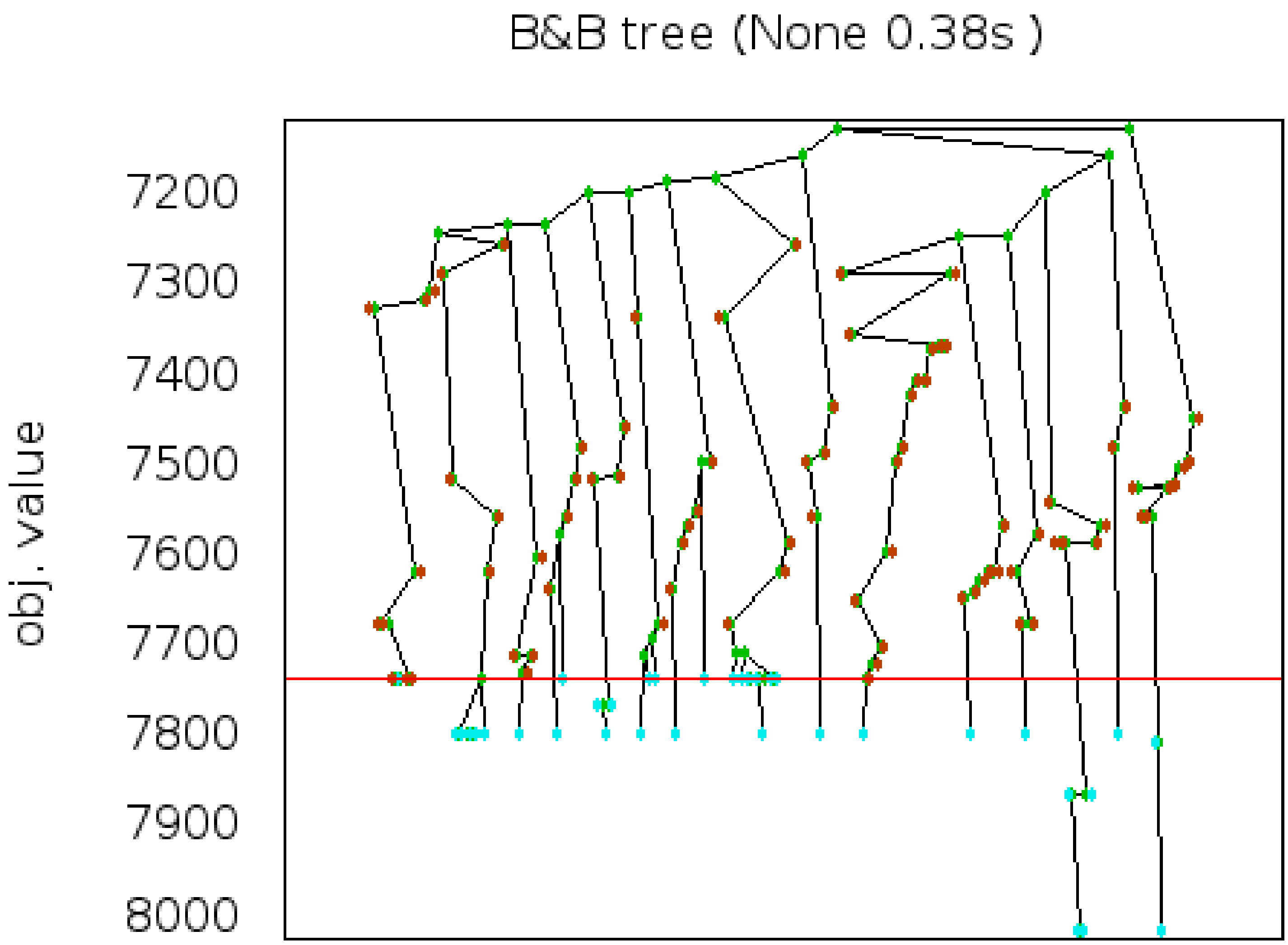


Figure 5: Tree after 400 nodes

Note that we are minimizing here!

A Thousand Words

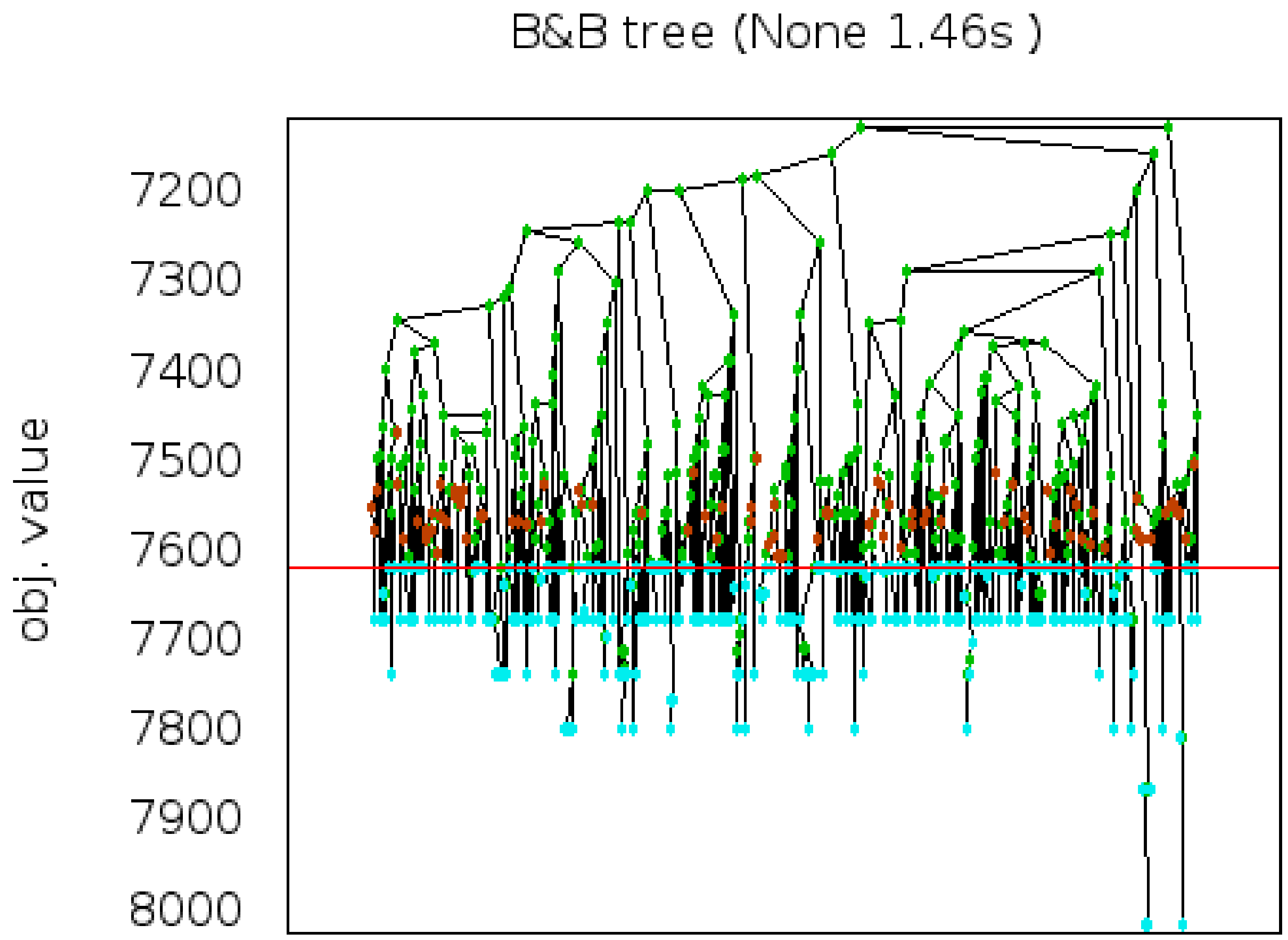


Figure 6: Tree after 1200 nodes

A Thousand Words

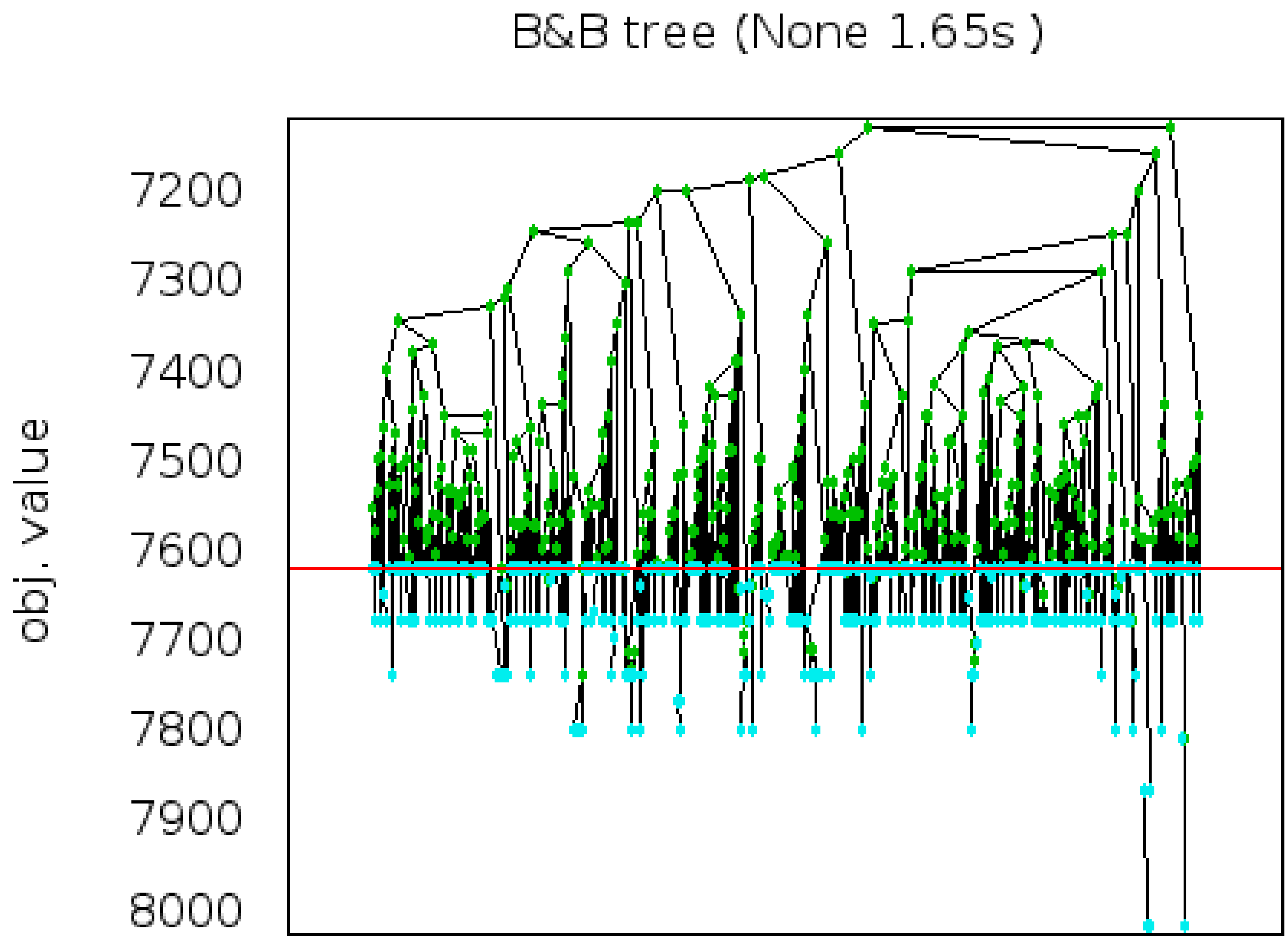


Figure 7: Final tree

Global Bounds

- The pictures show the evolution of the branch and bound process.
- Nodes are pictured at a height equal to that of their lower bound (we are minimizing in this case!!).
  - Red: candidates for processing/branching
  - Green: branched or infeasible
  - Turquoise: pruned by bound (possibly having produced a feasible solution) or infeasible.
- The red line is the level of the current best solution (global upper bound).
- The level of the highest red node is the global lower bound.
- As the procedure evolves, the two bounds grow together.
- The goal is for this to happen as quickly as possible.

Tradeoffs

- We will see that there are many tradeoffs to be managed in branch and bound.
- Note that in the final tree:
  - Nodes below the line were *pruned by bound* (and may or may not have generated a feasible solution) or were *infeasible*.
  - Nodes above the line were either *branched* or were *infeasible* or generated an *optimal solution*.
- There is a tradeoff between the goals of moving the upper and lower bounds
  - The nodes below the line serve to move the upper bound.
  - The nodes above the line serve to move the lower bound.
- It is clear that these two goals are somewhat antithetical.
- The search strategy has to achieve a balance between these two antithetical goals.

Tradeoffs in Practice

- In a practical implementation, there are many more choices and tradeoffs than those we have indicated so far.
- The complexity of the problem of optimizing the algorithm itself is immense.
- We have additional auxiliary methods, such as preprocessing and primal heuristics that we can choose to devote more or less effort to.
- We also have the choice of how much effort to devote to choosing a good candidate for branching.
- Finally, we have the choice of how much effort to devote to proving a good bound on the subproblem.
- It is the careful balance of the levels of effort devoted to each of these algorithmic processes the leads to a good algorithmic implementation.