Simple Example: Knapsack Problem

- We are given a set $N = \{1, \dots n\}$ of items and a capacity W.
- There is a profit p_i and a size w_i associated with each item $i \in N$.
- We want to choose the set of items that maximizes profit subject to the constraint that their total size does not exceed the capacity.
- The most straightforward formulation is to introduce a binary variable x_i associated with each item.
- x_i takes value 1 if item i is chosen and 0 otherwise.
- Then the formulation is

$$\begin{aligned} \max & \sum_{j=1}^{n} p_j x_j \\ \text{s.t.} & \sum_{j=1}^{n} w_j x_j \leq W \\ & x_i \in \{0,1\} \qquad \forall i \end{aligned}$$

- Is this formulation correct?

An Alternative Formulation

- Let us call a set $C \subseteq N$ a *cover* if $\sum_{i \in C} w_i > W$.
- Further, a cover C is *minimal* if $\sum_{i \in C \setminus \{j\}} w_i \leq W$ for all $j \in C$.
- Then we claim that the following is also a valid formulation of the original problem.

$$\begin{aligned} \max & \sum_{j=1}^{n} p_j x_j \\ \text{s.t.} & \sum_{j \in C} x_j \leq |C| - 1 \quad \text{for all minimal covers } C \\ & x_i \in \{0,1\} \qquad i \in N \end{aligned}$$

- Which formulation is "better"?